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M. VOORHOEVE

ANGULAR VARIATION AND THE ZEROS OF CERTAIN FUNCTIONS

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Angular variation and the zeros of certain functions*)

by

M. Voorhoeve

ABSTRACT

In this paper we study the zeros of functions

$$f(z) = \int_{\Gamma} e^{\zeta z} d\mu(\zeta),$$

where Γ is a Jordan arc and μ a complex measure on Γ.

Under certain conditions (i.e. if Γ is "almost a straight line"), we give explicit upper bounds for the number of zeros of f in a disk. The most important tool for proving this result is the concept of angular variation of a sequence developed in this paper.

KEY WORDS & PHRASES: *Zeros of analytic functions, Laplace transform, power sums*

*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In this paper we investigate the zeros of the Laplace transform of a measure μ, i.e. the function

$$(1.1) \qquad f(z) = \int_{\Gamma} e^{\zeta z} d\mu(\zeta),$$

where $\Gamma \subset \mathbb{C}$ is a Jordan arc and μ is a complex measure on Γ. Under certain specific conditions upon Γ and μ, we find an upper bound for the number of zeros of f in a disk around the origin. The resulting theorem and the method of its proof are best understood by following the order of the paper.

In Section 2 we define the *angle* of the complex numbers a,b as

$$A(a,b) = \begin{cases} |\mathrm{Arg}(b/a)| & \text{if } ab \neq 0 \quad , \\ \tfrac{1}{2}\pi & \text{if } ab = 0,\ a \neq b, \\ 0 & \text{if } a = b = 0 \quad . \end{cases}$$

We define the *angular variation* of a sequence $\{x_1,x_2,\ldots,x_n\}$ of complex numbers as

$$A(x_1,x_2,\ldots,x_n) = A(x_1,x_2) + A(x_2,x_3) + \ldots + A(x_{n-1},x_n).$$

We prove that the angular variation of a sequence is never inferior to the angular variation of the sequence of its partial sums.

In Section 3 we apply the results of the previous section to derive, by induction, a theorem on the moduli of the last terms of sequences that are obtained by iterated partial summation of a given sequence.

This result is then applied, in Sections 4 and 5, to power sums

$$s(k) = \sum_{\ell=1}^{N} C_\ell \omega_\ell^k,$$

where $C_1,\ldots,C_n$ and $\omega_1,\ldots,\omega_N$ are given complex numbers. Under the condition that

$$A(\omega_1-\omega_2,\omega_2-\omega_3,\ldots,\omega_{N-1}-\omega_N,\omega_N) < \tfrac{1}{2}\pi,$$

we derive constants n and u(t,k) such that for t = 1,2,...

$$|s(n+t)| \le \sum_{k=0}^{n} u(t,k)|s(k)|.$$

By limit transition, we apply this last result in Section 6 to the functions (1.1). Let $P = \{x_1,x_2,\dots,x_m\}$ be a partition of Γ. We define Γ_k as the section of Γ between x_{k-1} and x_k and set

$$\rho = \rho(\Gamma) = \sup_P A(x_1-x_2,x_2-x_3,\dots,x_{m-1}-x_m),$$

$$V(\Gamma,\mu) = \sup_P A(\mu(\Gamma_1),\dots,\mu(\Gamma_m)),$$

$$\lambda = \lambda(\Gamma) = \text{length of } \Gamma = \sup_P(|x_1-x_2| + \dots + |x_{m-1}-x_m|),$$

the suprema being taken over all partitions P of Γ. Under the conditions that $\rho(\Gamma) < \frac{1}{2}\pi$ and $V(\Gamma,\mu) < \infty$ we arrive at an estimate for the Taylor coefficients of f.

By standard techniques (see e.g. TIJDEMAN [1]) we then derive the following upper bound for the number $N_T(f)$ of zeros of f in a disk around the origin with radius T:

$$(1.2) \qquad N_T(f) \le 3n + 4T\lambda,$$

where n is the upper integer part of $\dfrac{V(\Gamma,\mu) - \frac{1}{2}\pi}{\frac{1}{2}\pi - \rho(\Gamma)}$.

2. ANGULAR VARIATION

A nonzero complex number ζ can be represented uniquely as

$$\zeta = re^{i\phi}; \quad r \ne 0, \quad -\pi < \phi \le \pi.$$

For such a $\zeta = re^{i\phi}$, we define $\mathrm{Arg}\zeta := \phi$. Let $a,b \in \mathbb{C}$. Then we define their angle A(a,b) by

$$A(a,b) = \begin{cases} |\mathrm{Arg}(b/a)| & \text{if } ab \neq 0 \quad , \\ \tfrac{1}{2}\pi & \text{if } ab = 0,\ a \neq b, \\ 0 & \text{if } a = b = 0 \quad . \end{cases}$$

Note that A(a,b) is a kind of distance function, satisfying

$$A(a,b) = A(b,a); \quad A(a,a) = 0; \quad A(a,b) + A(b,c) \geq A(a,c).$$

We also have

$$A(ac,bc) = A(a,b) \text{ if } c \neq 0; \qquad A(ac,bd) \leq A(a,b) + A(c,d).$$

LEMMA 2.1. *For* $a,b \in \mathbb{C}$

$$A(a,b) = A(a,a+b) + A(a+b,b).$$

PROOF. If a or b = 0, the proof follows immediately. The desired equation is symmetric in a and b, so we may assume without loss of generality that $\mathrm{Arg}(b/a) \geq 0$. For $x,y \in \mathbb{R}$ such that $x^2 + y^2 \neq 0$,

$$|\mathrm{Arg}(x+iy)| = \left| \int_0^{y/x} \frac{1}{1+t^2}\, dt \right|$$

decreases with increasing x if y is constant. Hence, since $\mathrm{Im}(b/a) \geq 0$,

$$0 \leq \mathrm{Arg}(1 + \frac{b}{a}) \leq \mathrm{Arg}(\frac{b}{a}),$$

so

$$\mathrm{Arg}(\frac{b}{a}) = \mathrm{Arg}(\frac{b+a}{a}) + \mathrm{Arg}(\frac{b}{b+a}),$$

proving the lemma. □

LEMMA 2.2. *Let* $a,b \in \mathbb{C}$, *such that* $|a| > |b|$. *Then*

$$A(a,b-a) > \tfrac{1}{2}\pi.$$

PROOF. Since $|b| < |a|$,

$$\operatorname{Re}(\frac{b}{a} - 1) = \operatorname{Re}(\frac{b}{a}) - 1 \leq |\frac{b}{a}| - 1 < 0,$$

so

$$A(a,b-a) = |\operatorname{Arg}(\frac{b}{a} - 1)| > \tfrac{1}{2}\pi. \qquad \square$$

For a sequence $S = \{x_1,x_2,\ldots,x_n\}$ of complex numbers we define its angular variation

$$A(S) = A(x_1,x_2,\ldots,x_n) = \sum_{k=1}^{n-1} A(x_k,x_{k+1}).$$

We have trivially

$$(2.1) \qquad A(x_1y_1,x_2y_2,\ldots,x_ny_n) \leq A(x_1,\ldots,x_n) + A(y_1,\ldots,y_n).$$

THEOREM 2.3. *Let* $S = \{x_1,x_2,\ldots,x_n\}$ *be a sequence of complex numbers and let* T *be the sequence of its partial sums* $x_1,x_1+x_2,\ldots,x_1+x_2+\ldots+x_n\}$. *Then*

$$A(T) \leq A(S) - A(x_1+x_2+\ldots+x_n,x_n).$$

PROOF. We apply induction on n. If $n = 1$, there is nothing to prove, so let $n > 1$. Let S' and T' be the sequences obtained from S and T respectively by deleting their last term. By the induction hypothesis

$$A(T) = A(T') + A(x_1+\ldots+x_{n-1},x_1+\ldots+x_n) \leq$$

$$\leq A(S') - A(x_1+\ldots+x_{n-1},x_{n-1}) + A(x_1+\ldots+x_{n-1},x_1+\ldots+x_n) =$$

$$= A(S) - A(x_{n-1},x_n) - A(x_1+\ldots+x_{n-1},x_{n-1}) + A(x_1+\ldots+x_{n-1},x_1+\ldots+x_n) \leq$$

$$\leq A(S) - A(x_1+\ldots+x_{n-1},x_n) + A(x_1+\ldots+x_{n-1},x_1+\ldots+x_n) =$$

$$= A(S) - A(x_1+x_2+\ldots+x_n,x_n),$$

by applying Lemma 2.1 with $a = x_1 + \ldots + x_{n-1}$, $b = x_n$. This proves our theorem. □

3. THE FUNDAMENTAL THEOREM

THEOREM 3.1. *For* $k = 1,2,\ldots$ *and* $j = 1,2,\ldots,N$, *let* $f_{k,j}$ *and* C_j *be complex numbers and denote* $M_k = \max_{j=1,\ldots,N} f_{k,j}$. *Define for* $m = 1,2,\ldots,N$,

$$s_0(m) = \sum_{j=1}^{m} C_j,$$

$$s_k(m) = \sum_{j=1}^{m} f_{k,j}s_{k-1}(j), \quad k = 1,2,\ldots .$$

For any $n \geq 0$ *such that*

$$A(C_1,\ldots,C_N) \leq \tfrac{1}{2}(n+1)\pi - \sum_{k=1}^{n} A(f_{k,1},\ldots,f_{k,N}), \tag{3.1}$$

we have for $t = 1,2,\ldots$

$$|s_{n+t}(N)| \leq \sum_{k=0}^{n} \binom{n+t-k-1}{n-k}\binom{N+n+t-k-1}{n+t-k} M_{n+t}\cdots M_{k+1}|s_k(N)|. \tag{3.2}$$

PROOF. We write

$$u_n(p,k,N) = \begin{cases} \binom{p-k-1}{n-k}\binom{N+p-k-1}{p-k} & \text{if } p > n,\ 0 \leq k \leq n, \\ 1 & \text{if } 0 \leq p = k \leq n \quad , \\ 0 & \text{otherwise} \end{cases}$$

and we shall prove by induction on N and p that - under the Condition (3.1) -

$$|s_p(N)| \leq \sum_{k=0}^{n} u_n(p,k,N)M_p\cdots M_{k+1}|s_k(N)|. \tag{3.3}$$

If $N = 1$ or if $p \leq n$, then (3.3) is trivial. So let $N > 1$, $p > n$. We have

$$|s_p(N)| = |s_p(N-1) + f_{p,N}s_{p-1}(N)| \leq |s_p(N-1)| + M_p|s_{p-1}(N)|.$$

By the induction hypothesis on p

$$(3.4)\qquad |s_{p-1}(N)| \le \sum_{k=0}^{n} u_n(p-1,k,n)M_{p-1}\cdots M_{k+1}|s_k(N)|.$$

Now suppose that $|s_0(N)| \ge |s_0(N-1)|$. Since (3.1) also holds if we replace N by N-1, we have by the induction hypothesis on N

$$|s_p(N-1)| \le \sum_{k=0}^{n} u_n(p,k,N-1)M_p\cdots M_{k+1}|s_k(N-1)| \le$$

$$\le \sum_{k=1}^{n} u_n(p,k,N-1)M_p\cdots M_{k+1}|s_k(N-1)| +$$

$$+ u_n(p,0,N-1)M_p\cdots M_1|s_0(N)|.$$

If $|s_0(N)| < |s_0(N-1)|$, then by Lemma 2.2

$$A(s_0(N-1),s_0(N) - s_0(N-1)) = A(s_0(N-1),C_N) > \tfrac{1}{2}\pi.$$

Hence by Theorem 2.3

$$A(s_0(1),\ldots,s_0(N-1)) \le A(C_1,\ldots,C_{N-1}) - A(s_0(N-1),C_{N-1}) =$$

$$(3.5)\qquad = A(C_1,\ldots,C_N) - A(C_{N-1},C_N) - A(s_0(N-1),C_{N-1})$$

$$< A(C_1,\ldots,C_N) - \tfrac{1}{2}\pi.$$

For m = 1,2,...,N-1, set

$$D_m = f_{1,m}s_0(m); \quad t_0(m) = \sum_{j=1}^{m} D_j; \quad t_k(m) = \sum_{j=1}^{m} f_{k-1,j}t_{k-1}(j).$$

Then by (2.1) and (3.5), applying (3.1),

$$A(D_1,\ldots,D_{N-1}) \le A(C_1,\ldots,C_N) - \tfrac{1}{2}\pi + A(f_{1,1},\ldots,f_{1,N-1}) \le$$

$$\le \tfrac{1}{2}n\pi - \sum_{k=2}^{n} A(f_{k,1},\ldots,f_{k,N}).$$

So the conditions of the theorem apply to the $t_k(m)$'s, with n replaced by n-1 and N by N-1. By the induction hypothesis on N we thus have

$$|s_p(N-1)| = |t_{p-1}(N-1)| \le \sum_{k=0}^{n-1} u_{n-1}(p-1,k,N-1)M_p \dots M_{k+2}|t_k(N-1)| =$$

$$= \sum_{k=1}^{n} u_n(p,k,N-1)M_p \dots M_{k+1}|s_k(N-1)|.$$

So either way we have

$$|s_p(N-1)| \le \sum_{k=1}^{n} u_n(p,k,N-1)M_p \dots M_{k+1}|s_k(N-1)| +$$

$$+ u_n(p,0,N-1)M_p \dots M_1|s_0(N)|.$$

Since for $k \ge 1$

$$|s_k(N-1)| \le |s_k(N)| + M_k|s_{k-1}(N)|,$$

we find

$$|s_p(N-1)| \le \sum_{k=0}^{n} (u_n(p,k,N-1) + u_n(p,k+1,N-1))M_p \dots M_{k+1}|s_k(N)|.$$

Hence, by (3.4),

$$|s_p(N)| \le \sum_{k=0}^{n} (u_n(p-1),k,N) + u_n(p,k,N-1) + u_n(p,k+1,N-1))M_p \dots$$
$$\dots M_{k+1}|s_k(N)|.$$

An easy calculation shows that

$$u_n(p-1,k,N) + u_n(p,k,N-1)) + u_n(p,k+1,N-1) \le u_n(p,k,N). \qquad \square$$

4. POWER SUMS

<u>THEOREM 4.1</u>. *Let $\omega_1,\omega_2,\dots,\omega_N$ and $C_1,\dots,C_N$ be given complex numbers. We define inductively*

$$s_0(m) = \sum_{k=1}^{m} c_k$$

$$s_\ell(m) = \sum_{k=1}^{m} s_{\ell-1}(k)(\omega_k - \omega_{k+\ell}) \qquad (\ell > 1),$$

where

$$\omega_{N+j} := 0 \textit{ if } j > 0.$$

Then

$$s_\ell(m) = \sum_{k=1}^{m} c_k(\omega_k - \omega_{m+1}) \cdots (\omega_k - \omega_{m+\ell}).$$

PROOF. We apply induction on ℓ. The theorem is trivial for $\ell = 0$. If $\ell > 0$, then by the induction hypothesis

$$s_\ell(m) = \sum_{k=1}^{m} (\omega_k - \omega_{k+\ell}) \sum_{\lambda=1}^{k} c_\lambda(\omega_\lambda - \omega_{k+1}) \ldots (\omega_\lambda - \omega_{k+\ell-1}) =$$

$$= \sum_{\lambda=1}^{m} c_\lambda \sum_{k=\lambda}^{m} (\omega_\lambda - \omega_{k+1})(\omega_\lambda - \omega_{k+2}) \ldots (\omega_\lambda - \omega_{k+\ell-1})(\omega_k - \omega_{k+\ell}).$$

It is thus sufficient to prove that

$$\sum_{k=\lambda}^{m} (\omega_\lambda - \omega_{k+1}) \ldots (\omega_\lambda - \omega_{k+\ell-1})(\omega_k - \omega_{k+\ell}) = (\omega_\lambda - \omega_{m+1}) \ldots (\omega_\lambda - \omega_{m+\ell}). \tag{4.1}$$

This we prove by induction on m. If $m = \lambda$, then (4.1) follows immediately. If $m > \lambda$, then by the induction hypothesis

$$\sum_{k=\lambda}^{m} (\omega_\lambda - \omega_{k+1}) \ldots (\omega_\lambda - \omega_{k+\ell-1})(\omega_k - \omega_{k+\ell}) =$$

$$= (\omega_\lambda - \omega_m)(\omega_\lambda - \omega_{m+1}) \ldots (\omega_\lambda - \omega_{m+\ell-1}) + (\omega_\lambda - \omega_{m+1}) \ldots (\omega_\lambda - \omega_{m+\ell-1})(\omega_m - \omega_{m+\ell}) =$$

$$= (\omega_\lambda - \omega_{m+1}) \ldots (\omega_\lambda - \omega_{m+\ell-1})(\omega_\lambda - \omega_{m+\ell}),$$

proving (4.1) and the theorem. □

COROLLARY 4.2. *In the notation of Theorem* 4.1, $s_\ell(N) = \sum_{k=1}^{N} c_k \omega_k^\ell$.

Corollary 4.2 shows that Theorem 3.1 can be applied to power sums, with $f_{\ell,k} = \omega_k-\omega_{k+\ell}$. So we shall investigate the angular variation of the sequences $\{\omega_1-\omega_{\ell+1},\omega_2-\omega_{\ell+2},\ldots,\omega_N-\omega_{\ell+N}\}$.

<u>LEMMA 4.3</u>. *Let* $a,b,c,d \in \mathbb{C}$ *such that* $A(a-b,b-c,c-d) \leq \pi$. *Then*

$$(4.2) \qquad A(a-c,b-c,b-d) \geq A(a-c,a-d,b-d).$$

<u>PROOF</u>. If $a = b$, then (4.2) is trivial. If $b = c$, then by Lemma 2.1,

$$A(a-c,a-d,b-d) = A(a-b,a-d,b-d) = A(a-b,b-d) \leq A(a-c,b-c,b-d)$$

by the triangle inequality, proving (4.2). If $a = c$, then $A(a-b,b-c,c-d) \leq \pi$ implies that $A(b-c,c-d) = A(b-d,c-d) = 0$, so (4.2) holds. If $a = d$, by Lemma 2.1

$$\pi \geq A(a-b,b-c,c-d) = A(a-b,a-c,b-c,c-a),$$

so $A(a-b,a-c) = 0$. Hence

$$A(a-c,b-c,b-a) \geq A(a-c,b-a) = \pi = A(a-c,0,b-a),$$

if $b \neq a$ and $c \neq a$, proving (4.2). Since the lemma is not alterated by the transformation $T: (a,b,c,d) \to (d,c,b,a)$, we may thus assume that a,b,c and d are all different.

If $A(a-b,b-c,c-d) = A(a-b,c-d)$, then by repeated application of Lemma 2.1

$$A(a-b,c-d) = A(a-b,a-c,b-c,c-d) = A(a-b,a-c,c-d) =$$

$$= A(a-b,a-c,a-d,c-d).$$

Likewise, $A(a-b,c-d) = A(a-b,a-d,b-d,c-d)$, so

$$A(a-b,c-d) = A(a-b,a-c,a-d,b-d,c-d) = A(a-b,b-c,c-d) =$$

$$= A(a-b,a-c,b-c,b-d,c-d).$$

Hence $A(a-c,a-d,b-d) = A(a-c,b-c,b-d)$, proving the lemma. So we may assume that $A(a-c,b-c,c-d) > A(a-b,c-d)$. Without loss of generality we may assume that

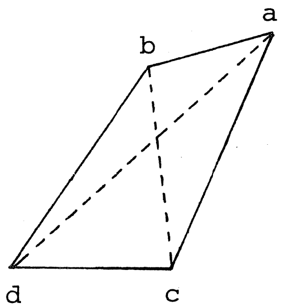
$$A(b-c,c-d) \geq A(a-b,b-c);\ \mathrm{Arg}(c-d) = 0;\ \mathrm{Arg}(b-c) \geq 0,$$

since we can apply T, multiply by a nonzero constant and take conjugates everywhere without altering the lemma. So, again applying Lemma 2.1,

$$0 \leq \mathrm{Arg}(a-b) \leq \mathrm{Arg}(a-c) \leq \mathrm{Arg}(b-c),$$

$$0 = \mathrm{Arg}(c-d) \leq \mathrm{Arg}(a-d) \leq \mathrm{Arg}(a-c) \leq \mathrm{Arg}(b-c).$$

If $A(a-c,a-d,b-d) = A(a-c,b-d)$, then (4.2) immediately follows, so we may assume that $\mathrm{Arg}(b-d) > \mathrm{Arg}(a-d)$. The points d, c, a and b then form a quadrangle as in the figure.



Since $A(b-d,c-d) + A(a-b,a-c) \leq A(a-b,b-c,c-d) \leq \pi$, the point a is exterior to or on the circle through b, c and d. Hence $A(a-c,a-d) \leq A(b-c,b-d)$. Similarly $A(a-d,b-d) \leq A(a-c,b-c)$. This proves the lemma. □

<u>THEOREM 4.4</u>. *Let* $\omega_1,\omega_2,\ldots,\omega_N$ *be given complex numbers, and for* $k = 1,2,\ldots$ *let* T_k *be the sequence* $T_k = \{\omega_1-\omega_{k+1},\omega_1-\omega_{k+2},\omega_2-\omega_{k+2},\omega_2-\omega_{k+3},\ldots$ $\ldots,\omega_{N-1}-\omega_{k+N},\omega_N-\omega_{k+N}\}$, *where* $\omega_j := 0$ *if* $j > N$. *If* $A(T_1) \leq \pi$, *then for* $k = 1,2,\ldots$

$$A(T_{k+1}) \leq A(T_k) - A(\omega_1-\omega_{k+1},\omega_1-\omega_{k+2}).$$

<u>PROOF</u>. Let S_k be the sequence obtained from T_k by omitting the first term $\omega_1-\omega_{k+1}$ and adding a final term $\omega_N-\omega_{k+N+1}$. Since $A(\omega_N-\omega_{k+N},\omega_N-\omega_{k+N+1}) = 0$ if $k \geq 1$, we have

$$A(S_k) = A(T_k) - A(\omega_1-\omega_{k+1},\omega_1-\omega_{k+2}).$$

Now T_{k+1} is obtained from S_k by changing each term $\omega_\ell-\omega_{k+\ell}$ in S_k into $\omega_{\ell-1}-\omega_{k+\ell+1}$ for $\ell = 2,3,\ldots,N$. Since for $\ell = 2,3,\ldots,N-k$, by repeated application of Lemma 2.1,

$$\pi \geq A(T_1) \geq A(\omega_{\ell-1}-\omega_\ell,\omega_\ell-\omega_{\ell+1},\omega_{\ell+1}-\omega_{\ell+2},\ldots,\omega_{\ell+k}-\omega_{\ell+k+1}) \geq$$

$$\geq A(\omega_{\ell-1}-\omega_\ell,\omega_\ell-\omega_{\ell+2},\omega_{\ell+2}-\omega_{\ell+3},\ldots,\omega_{\ell+k}-\omega_{\ell+k+1}) \geq \ldots \geq$$

$$\geq A(\omega_{\ell-1}-\omega_\ell,\omega_\ell-\omega_{\ell+k},\omega_{\ell+k}-\omega_{\ell+k+1}),$$

we have for $\ell = 2,3,\ldots,N-k$ by Lemma 4.3,

$$A(\omega_{\ell-1}-\omega_{k+\ell},\omega_\ell-\omega_{k+\ell},\omega_\ell-\omega_{k+\ell+1}) \geq$$

$$\geq A(\omega_{\ell-1}-\omega_{k+\ell},\omega_{\ell-1}-\omega_{k+\ell+1},\omega_\ell-\omega_{k+\ell+1}).$$

For $\ell > N-k$ the same inequality trivially holds. So

$$A(T_{k+1}) \leq A(S_k) = A(T_k) - A(\omega_1-\omega_{k+1},\omega_1-\omega_{k+2}),$$

proving the theorem. □

<u>COROLLARY 4.5</u>. *If* $A(\omega_1-\omega_2,\omega_2-\omega_3,\ldots,\omega_N-\omega_{N-1}) \leq \pi$, *then for* $k = 1,2,\ldots,$

$$A(\omega_1-\omega_{k+1},\omega_2-\omega_{k+2},\ldots,\omega_N-\omega_{k+N}) \leq A(\omega_1-\omega_2,\omega_2-\omega_3,\ldots,\omega_N-\omega_{N+1}).$$

<u>PROOF</u>. The sequence $S_k' = \{\omega_1-\omega_{k+1},\omega_2-\omega_{k+2},\ldots,\omega_N-\omega_{k+N}\}$ is a subsequence of T_k, so by Theorem 4.4, if $A(T_1) \leq \pi$, then

$$A(S_k') \leq A(T_k) \leq A(T_1).$$

By Lemma 2.2

$$A(T_1) = A(\omega_1-\omega_2,\omega_1-\omega_3,\omega_2-\omega_3,\ldots,\omega_{N-1}-\omega_{N+1},\omega_N-\omega_{N+1}) =$$

$$= A(\omega_1-\omega_2,\omega_2-\omega_3,\ldots,\omega_N-\omega_{N+1}) \leq \pi.$$

This proves the corollary. □

5. THE MAIN THEOREM

THEOREM 5.1. *Let* $C_1,\ldots,C_N$ *and* $\omega_1,\ldots,\omega_N$ *be given complex numbers, such that*

$$A(\omega_1-\omega_2,\omega_2-\omega_3,\ldots,\omega_{N-1}-\omega_N,\omega_N) = \rho < \tfrac{1}{2}\pi.$$

Define

$$\lambda = |\omega_1-\omega_2| + \ldots + |\omega_{N-1}-\omega_N| + |\omega_N|;$$

$$n = \max\left(0, \left\lceil \frac{A(C_1,\ldots,C_N) - \frac{1}{2}\pi}{\frac{1}{2}\pi - \rho} \right\rceil\right),$$

where $\lceil x \rceil$ *denotes the smallest integer* m *such that* $m \geq x$. *Then for* $t = 1,2,\ldots,$ *the power sums* $s(k) = \sum_{\ell=1}^{N} C_\ell \omega_\ell^k$ *satisfy*

$$|s(n+t)| \leq \sum_{k=0}^{n} \binom{n+t-k-1}{n-k}\binom{n+t}{k}\lambda^{n+t-k}|s(k)|.$$

PROOF. Let $\mu := \max_{k=1,\ldots,N} |\omega_k-\omega_{k+1}|$, where we define $\omega_{N+1} := 0$. Set $f_{\ell,k} = \omega_k-\omega_{k+\ell}$. By Corollary 4.2, $s(k) = s_k(N)$ in the notation of Theorem 4.1. By Corollary 4.5

$$A(f_{\ell,1},\ldots,f_{\ell,N}) \leq \rho,$$

thus

$$A(C_1,\ldots,C_N) \leq \tfrac{1}{2}\pi + \sum_{\ell=1}^{n} \{\tfrac{1}{2}\pi - A(f_{\ell,1},\ldots,f_{\ell,N})\},$$

so we can apply Theorem 3.1. Since for $\ell = 1,2,\ldots$

$$M_\ell = \max_{k=1,\ldots,N} |\omega_k-\omega_{k+\ell}| \leq \ell\mu,$$

we have by Theorem 3.1

$$(5.1) \qquad |s(n+t)| \leq \sum_{k=0}^{n} (n+t)\ldots(k+1)\binom{n+t-k-1}{n-k}\binom{N+n+t-k-1}{n+t-k}\mu^{n+t-k}|s(k)|.$$

This estimate can be improved by choosing equidistant points $\alpha_{i,1},\ldots,\alpha_{i,t(i)}$ on the line segment between ω_i and ω_{i+1} in such a way that the total number M of points is very large and

$$\max_{i=1,\dots,n} \{ \max_{\lambda=1,\dots,t(i)} (|\alpha_{i,1}-\omega_i|, |\alpha_{i,\lambda+1}-\alpha_{i,\lambda}|, |\omega_{i+1}-\alpha_{i,t(i)}|)\} \le$$

$$\le \frac{\lambda(1+\varepsilon)}{M}$$

for a small $\varepsilon > 0$. We now form for $\delta > 0$ the power sums

$$s_\delta(k) = \sum_{i=1}^{N} C_i\left(\omega_i^k + \sum_{\lambda=1}^{t(i)} \delta\, \alpha_{i,\lambda}^k\right).$$

Since $A(\omega_i-\alpha_{i,1},\dots,\alpha_{i,t(i)}-\omega_{i+1}) = A(\omega_i,\omega_{i+1})$ and $A(C_i,C_{i+1}) =$ $= A(C_i,\delta C_i,\dots,\delta C_i,C_{i+1})$, we have by (5.1)

$$|s_\delta(n+t)| \le \sum_{k=0}^{n} \binom{n+t-k-1}{n-k} \cdot \frac{(n+t)!}{k!} \cdot \binom{M+n+t-k-1}{n+t-k}\left(\frac{\lambda(1+\varepsilon)}{M}\right)^{n+t-k} |s_\delta(k)|.$$

Now we let $M \to \infty$ and $\varepsilon \downarrow 0$, giving

$$|s_\delta(n+t)| \le \sum_{k=0}^{n} \binom{n+t-k-1}{n-k}\binom{n+t}{k}\lambda^{n+t-k} |s_\delta(k)|.$$

By letting $\delta \downarrow 0$ the theorem is proved. □

6. LAPLACE TRANSFORMS

Let Γ be a Jordan arc in $\mathbb{C}$ of finite length running from a to b. Let μ be a complex measure on Γ, that is a combination $\mu_1 + i\mu_2 - \mu_3 - i\mu_4$ of four positive measures on Γ. We consider not identically vanishing functions of the form (1.1):

$$f(z) = \int_\Gamma e^{\zeta z} d\mu(\zeta).$$

In studying the zeros of f, we may assume without loss of generality that $b = 0$, since we can multiply f by e^{-bz} without altering its zeros. Now let

$$P = \{a = x_1, x_2, \dots, x_m = b\}$$

be a partition of Γ and, for $k = 1,2,\dots,m-1$, let Γ_k be the segment of Γ

running from x_k to x_{k+1}. We set

$$M(P) = \max_{k=1,\dots,m-1} |x_k - x_{k-1}|.$$

We now construct the exponential polynomials

$$F_P(z) = \sum_{k=1}^{m-1} \exp(x_k z)\mu(\Gamma_k).$$

<u>LEMMA 6.1</u>. *Let* $D \subset \mathbb{C}$ *be bounded and let* $P_1, P_2, \dots$ *be a sequence of partitions such that* $\lim_{k\to\infty} M(P_k) = 0$. *Then*

$$\lim_{k\to\infty} F_{P_k}(z) = f(z),$$

uniformly for $z \in D$.

<u>PROOF</u>. Observe that for a partition $P = \{x_1, x_2, \dots, x_m\}$

$$|F_P(z) - f(z)| = \left|\sum_{k=1}^{m-1} \int_{\Gamma_k} (\exp(x_k z) - \exp(\zeta z))d\mu(\zeta)\right| \leq$$

$$\leq \sum_{j=1}^{4} \mu_j(\Gamma) \cdot \max_{k=1,\dots,m-1} \max_{\zeta\in\Gamma_k} |\exp(x_k z) - \exp(\zeta z)|.$$

For $z \in D$,

$$|\exp(x_k z) - \exp(\zeta z)| \to 0$$

uniformly if $\max|x_k - \zeta| \to 0$. □

We consider curves Γ with the property that $b = 0$ and

$$\rho(\Gamma) := \sup_P\{A(x_1-x_2, x_2-x_3, \dots, x_{m-1}-x_m)\} < \tfrac{1}{2}\pi,$$

the supremum being taken over all partitions $P = \{a = x_1, x_2, \dots, x_m = 0\}$ of Γ. We define

$$V(\Gamma,\mu) = \sup_P A(\mu(\Gamma_1, \dots, \mu(\Gamma_m))$$

and set

$$(6.1)\qquad n = \left\lceil \frac{V(\Gamma,\mu) - \frac{1}{2}\pi}{\frac{1}{2}\pi - \rho(\Gamma)} \right\rceil .$$

Write $\lambda = \lambda(\Gamma) = \sup_P(|x_1-x_2|+\ldots+|x_{m-1}-x_m|)$, taken over all partitions $P = \{x_1,\ldots,x_m\}$ of Γ. This is the length of Γ. By Theorem 5.1, since $x_m = b = 0$ and since for $\ell = 0,1,\ldots,$

$$F_P^{(\ell)}(0) = \sum_{k=1}^{m-1} \mu(\Gamma_k)x_k^{\ell},$$

we now have

$$(6.2)\qquad |F_P^{(n+t)}(0)| \le \sum_{\ell=0}^{n} \binom{n+t-\ell-1}{n-\ell}\binom{n+t}{\ell}\lambda^{n+t-\ell}|F_P^{(\ell)}(0)|.$$

Writing $f(z) = \sum_{k=0}^{\infty} \phi_k z^k$, and observing that, by Lemma 6.1, $\phi_k = f^{(k)}(0)/k!$ is approximated by $F_{P_m}^{(k)}(0)/k!$, we find from (6.2) for $t = 1,2,\ldots,$

$$(6.3)\qquad |\phi_{n+t}| \le \sum_{\ell=0}^{n} \frac{\lambda^{n+t-\ell}}{(n-\ell)!(t-1)!(n+t-\ell)} |\phi_\ell| .$$

Hence

$$\sum_{t=1}^{\infty} |\phi_{n+t}|R^{n+t} \le (R/r)^n \sum_{t=1}^{\infty} (R\lambda)^t/t! \sum_{\ell=0}^{n} \frac{(\lambda r)^{n-\ell}}{(n-\ell)!} |\phi_\ell|r^\ell \le$$

$$\le (R/r)^n(e^{\lambda R}-1)e^{\lambda r} \max_{|z|\le r} |f(z)| .$$

Moreover,

$$\sum_{\ell=0}^{n} |\phi_\ell|R^\ell \le (R/r)^n(1+(r/R)+\ldots+(r/R)^n) \max_{\ell=0,\ldots,n} (|\phi_\ell|r^\ell) \le$$

$$\le (R/r)^n\cdot(R/R-r))\cdot \max_{|z|\le r} |f(z)| .$$

Hence,

$$(6.4)\qquad \max_{|z|\le R} |f(z)| < (R/r)^n e^{\lambda R} \max\{e^{\lambda R},R/(R-r)\} \max_{z\le r} |f(z)| . \qquad \square$$

The following lemma is more or less standard.

LEMMA 6.2. *Let* r, R *and* T *be real numbers satisfying* $R > r > 0$; $R \geq T > 0$. *If* $f \not\equiv 0$ *is analytic in a neighbourhood of the disk* $|z| \leq R$, *then the number* $N_T(f)$ *of zeros of* f *in the disk* $\{z \in \mathbb{C};\ |z| \leq T\}$, *counted according to multiplicity, satisfies*

$$N_T(f)\log\left(\frac{R^2-rT}{R(T+r)}\right) \leq \log\{\max_{|z|\leq R}|f(z)| \Big/ \max_{|z|\leq r}|f(z)|\}.$$

PROOF. See WALDSCHMIDT [4], page 6.4, Lemma 6.2.1. □

For other versions of Lemma 6.2, compare TIJDEMAN [1], Lemma 1 and VOORHOEVE [2], Lemma 4.4. We can now prove our final theorem on the zeros of the Laplace transform (1.1):

THEOREM 6.3. *For functions* $f \not\equiv 0$ *defined by* (1.1) *we have*

$$N_T(f) \leq 3n + 4T\lambda, \tag{6.5}$$

where n *is defined by* (6.1) *and* λ *is the length of* Γ.

PROOF. Apply Lemma 6.2 with $R = 4T$ and $r = T/10$. Inserting (6.4), we obtain

$$N_T(f)\log(159/44) \leq n\log 40 + 4\lambda T + \max\{\lambda T/10, \log(40/39)\}.$$

This yields (6.5) if $n > 0$ or if $\lambda T > 1/5$. If $n = 0$ and $\lambda T \leq 1/5$, observe that $N_T(f) \leq 4/5 + \log(40/39) < 1$, so $N_T(f) = 0$, proving (6.5) completely. □

We can use Theorem 6.3 to estimate the number of zeros of f in a disk with centre $c \neq 0$, either directly by estimating $N_{T+|c|}(f)$, or indirectly by observing that

$$\tilde{f}(z) = f(z+c) = \int_\Gamma e^{\zeta z} d\tilde{\mu}(\zeta),$$

where $\tilde{\mu}$ is defined by $\tilde{\mu}(\gamma) = \int_\gamma e^{c\zeta} d\mu(\zeta)$ for a measurable $\gamma \subset \Gamma$. So

$$N_T(\tilde{f}) \leq 3\tilde{n} + 4\lambda T, \tag{6.6}$$

where

$$\tilde{n} = \left\lceil \frac{V(\Gamma,\tilde{\mu}) - \frac{1}{2}\pi}{\frac{1}{2}\pi - \rho} \right\rceil .$$

One can estimate $V(\Gamma,\tilde{\mu})$ by

$$V(\Gamma,\tilde{\mu}) \le V(\Gamma,\mu) + |c|\lambda,$$

so $\tilde{n} \le n + \lceil |c|\lambda/(\frac{1}{2}\pi-\rho) \rceil$. This rough estimate for $\tilde{n}$, inserted in (6.4), does not give an essentially better result than the direct approach. In some cases, though, one can give better estimates for $\tilde{n}$.

7. REMARKS

1. If Γ is a straight line segment (i.e. $\rho(\Gamma) = 0$, then a more direct approach in estimating the zeros of f is possible; see VOORHOEVE [3]. Also if $V(\Gamma,\mu) < \pi$, it is easy to show that $f(0) \neq 0$ and it is not hard to give estimates of the form $|f^{(t)}(0)| \le u(t)|f(0)|$ for $t = 1,2,\ldots$.

2. It is clear that the dependency on T and λ in the estimate (6.5) is right - that is, up to a constant factor. For instance we can take

$$f(z) = 1 + e^{\lambda z} = \int_\Gamma e^{\zeta z} d\mu(\zeta),$$

   where $a = \lambda$, $b = 0$ and $\mu = 0$ except for two point masses at a and b. Then $N_T(f) \sim \lambda T/(2\pi)$. However, it may be possible to improve the constant n and the condition that $\rho(\Gamma) < \frac{1}{2}\pi$.

3. By this method, one cannot derive an analogue of Theorem 6.3 for all curves Γ with $\rho(\Gamma) > \pi$, because such curves can be closed. For closed curves Γ, we can derive counterexamples to such a theorem, using Cauchy's theorem

$$f(z) = \int_\Gamma e^{\zeta z} d\mu(\zeta) = \int_\Gamma e^{\zeta z} d(\mu(\zeta) + C\mu_0(\zeta)),$$

   where μ_0 is the Lebesgue measure. By letting $C \to \infty$, the $V(\Gamma,\mu + C\mu_0)$ approximate $V(\Gamma,\mu_0)$. So this would imply that there is an absolute upper bound for $N_T(f)$, independent of μ. By a suitable choice of μ, however, $f(z)$ can have zeros of arbitrary multiplicity. This is a contradiction.

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